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FAST TRACK COMMUNICATION

Markov-breaking and the emergence of long memory in Ornstein–Uhlenbeck systems

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Abstract
We consider a complex system composed of many non-identical parts where (i) the dynamics of each part are Ornstein–Uhlenbeck; (ii) all parts are driven by a common external Lévy noise; and (iii) the system's collective output is the averaged aggregate of the outputs of its parts. Whereas the dynamics on the 'microscopic' parts-level are Markov, the dynamics on the 'macroscopic' system-level are not Markov—and may display a long memory. Moreover, the universal temporal scaling limit of the system's output, in the presence of long memory, is fractional Brownian motion. The model presented is analytically tractable, and gives closed-form quantitative characterizations of both the Markov-breaking phenomenon and the emergence of long memory.

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1. Introduction

In this communication we show how *fractional Brownian motion* (FBM) arises naturally from complex systems whose parts are governed by *Ornstein–Uhlenbeck* (OU) dynamics driven by a common external *Lévy noise*, and whose output is the averaged aggregate of their parts' outputs.

OU dynamics are the simplest conceivable stochastic dynamics in continuous time, and are both elemental and ubiquitous in the physical and engineering sciences [1, 2]. FBM, on the other hand, is a highly complex random motion, and its intrinsic structure is highly non-intuitive and intricate [3, 4].

In physics, FBM was shown to emerge from Hamiltonian dynamics leading to generalized Langevin equations: (i) heat baths with random–matrix interactions [5]; (ii) Kac–Zwanzig heat baths with random initial conditions [6]. In mathematics, abstract limit theorems leading to FBM were devised (see [4, 7] and references therein), and independent superpositions of simple random processes were proved to converge to FBM: (i) renewal processes [8, 9]; (ii) on–off processes [10]; (iii) persistent random walks [11]; (iv) OU processes [12].

As observed by Hurst [13], the Nile basin is a complex hydrological system producing FBM. The Nile basin is composed of many different geological parts. All parts are 'fed' by rainfall, each part produces its output flow, and the parts' output flows are aggregated-up into the Nile flow. The Nile basin gives rise to the following conceptual model: the rainfall events are a common external Poissonian noise affecting all parts, and the parts' output flows are (non-identical) shot noise processes driven by the common 'rainfall noise'[3].

Since shot noise is a special case of Lévy-driven OU dynamics [14–16], we arrive at the *Composite Ornstein–Uhlenbeck* (COU) system-model presented and studied in this communication: a system whose output is the averaged aggregate of the outputs of its many different system parts—each part governed by its own OU dynamics, and all dynamics driven by a common external Lévy noise.

The COU system-model establishes a path leading from the elemental OU dynamics to the complex FBM. The path will take us through moving-average processes, the phenomena of Markov-breaking and the emergence of long memory.

2. Ornstein–Uhlenbeck dynamics

OU dynamics [1] are governed by the linear stochastic differential equation

$$\dot{\xi}(t) = -x\xi(t) + y\dot{N}(t) \tag{1}$$

where (i) the 'input' $\dot{N} = (\dot{N}(t))_t$ is a driving noise process; (ii) the 'output' $\xi = (\xi(t))_t$ is the OU process propagated by the OU dynamics; (iii) the 'amplitudes' x and y are arbitrary positive parameters. The general solution of the OU equation (1), over the entire real line $-\infty < t < \infty$, is given by

$$\xi(t) = \int_{-\infty}^{t} (y\exp\{-x\cdot(t-t')\})\dot{N}(t')\,\mathrm{d}t'. \tag{2}$$

Equation (2) represents a linear integral transformation mapping the input noise process $\dot{N}$ to the output OU process ξ.

The OU equation (1) is a *Langevin* equation with a linear restoring force. It describes a system undergoing an exponential relaxation while, simultaneously, being perturbed by an external noise. The exponential relaxation and the perturbing noise are antithetical—the former pushing the system toward equilibrium, while the latter driving the system away from equilibrium. The parameter x represents the amplitude of the exponential relaxation, whereas the parameter y represents the amplitude of the perturbing noise.

In case the driving noise is *white*—the temporal derivative of *Brownian motion*—the OU equation (1) describes *diffusion* in the presence of a harmonic potential well [1, 2] . In case the driving noise is *Poissonian*—the temporal derivative of a *compound Poisson process*—the OU equation (1) describes *shot noise* (see [15, 16] for a modern review).

Both white and Poissonian noises are special cases of *Lévy noises*—the temporal derivatives of *Lévy processes* (namely, processes with stationary and independent increments [17, 18]). OU and Langevin dynamics driven by general Lévy noises attracted major interest in recent years, and were studied via different perspectives and approaches [14, 19–28].

[3] The Nile example was chosen due to the connection to Hurst. Since the Nile basin is vast, it is not realistic to assume the rainfall over its different parts to be modeled by one common external process. A more realistic example is that of a river basin 'fed' by common rainfall events, and composed of many water catchments assuming the role of its different parts.

3. Composite Ornstein–Uhlenbeck systems

The COU system-model is constructed as follows. Consider a system composed of n non-identical parts labeled $k = 1, \ldots, n$. Each part is governed by its own OU dynamics, and all parts are driven by a common noise $\dot{N}$:

$$\dot{\xi}_k(t) = -X_k \xi_k(t) + Y_k \dot{N}(t), \tag{3}$$

where $\xi_k = (\xi_k(t))_t$ and (X_k, Y_k) are, respectively, the OU output process and the 'amplitude-pair' of part k. The system's output process $\Xi_n = (\Xi_n(t))_t$ is the averaged aggregate of the output processes of its parts:

$$\Xi_n(t) = \frac{1}{n} \sum_{k=1}^{n} \xi_k(t). \tag{4}$$

We emphasize that in the COU system-model the output processes of the system-parts are coupled via the common driving noise $\dot{N}$. Superpositions of independent OU processes—where each OU process is driven by its own noise process, and the driving noises are independent—were explored in [12] and in [29, 30] (see also references therein).

The COU system may be considered as a conceptual hydrological model of river flows. The external noise $\dot{N}$ represents the rainfall over the river basin—the rainfall events modeled in the form of Poissonian 'shots'. The river basin is composed of many different water catchments. Each water catchment (k) produces its own output process (ξ_k) which is a shot noise process driven by the 'rainfall noise' $\dot{N}$. The outputs of all water catchments are aggregated together into the river flow $\left(\sum_{k=1}^{n} \xi_k\right)$.

We focus on the case of large COU systems in which $n \gg 1$. In this case it is natural to assume that the variability of the amplitude-pairs $\{(X_k, Y_k)\}_{k=1}^{n}$ obeys some statistical regularity. Specifically, we assume that the amplitude-pairs are (i) independent and identically distributed copies of a random, non-negative valued, amplitude-pair (X, Y); (ii) independent of the driving noise $\dot{N}$. We emphasize that—in accordance with the fluctuation–dissipation theorem [31]—the amplitudes X and Y may certainly be dependent random variables. The input noise $\dot{N}$ is considered Lévy.

Thus, the COU system-model has two sources of underlying randomness: '*quenched*' and '*annealed*'. The quenched randomness is internal and static: it influences the system at its formation epoch by 'molding', once and for all, the realizations of the parts' amplitude-pairs (X_k, Y_k). The annealed randomness is external and dynamic: it influences the system after its formation, perturbing it via the external driving noise $\dot{N}$.

As the system-size grows to infinity ($n \to \infty$), the system's output process Ξ_n converges stochastically to a limit process $\Xi = (\Xi(t))_t$ given by

$$\Xi(t) = \int_{-\infty}^{t} \Phi(t - t')\dot{N}(t')\,\mathrm{d}t', \tag{5}$$

where $\Phi(\tau) = \langle Y \exp\{-\tau X\}\rangle$ $(\tau > 0)$ [32]. The process Ξ is a temporal *moving-average* (MA) of the driving noise $\dot{N}$, and the temporal averaging is performed with respect to the *memory function* $\Phi(\tau)$—which is the mean of the random variable $Y \exp\{-\tau X\}$. The stochastic convergence requires that the random ratio Y^2/X have a finite mean, and that the driving Lévy noise have a finite variance.

(The stochastic convergence $\Xi_n \to \Xi$ is in L^2, and its proof is based on L^2 techniques. The convergence holds even in cases where the amplitude-pairs are dependent random variables—provided that their correlations are not too 'strong' [32].)

Equation (5) represents a linear integral transformation mapping the input noise process $\dot{N}$ to the output MA process Ξ, and is a generalization of equation (2)—replacing the exponential memory function (of equation (2)) by the general memory function $\Phi(\tau)$.

If the memory function $\Phi(\tau)$ is exponential—a degenerate scenario taking place if and only if the amplitude X is non-random—then differentiating both sides of equation (5) yields back the OU stochastic differential equation (1). However, if the memory function $\Phi(\tau)$ is not exponential then differentiating repeatedly both sides of equation (5) yields an infinite cascade of stochastic differential equations:

$$\dot{U}_m(t) = U_{m+1}(t) + \kappa_m \dot{N}(t) \tag{6}$$

($m = 0, 1, 2, \ldots$), where $U_0(t) = \Xi(t)$ and $\kappa_m = (-1)^m \langle X^m Y \rangle$. Thus, shifting from exponential to non-exponential memory functions in the MA dynamics of equation (5) results in an 'explosion of dimensionality'—the one-dimensional OU stochastic differential equation (1) being replaced by the infinite-dimensional system of stochastic differential equations (6).

4. Markov-breaking and long memory

In the case of OU dynamics a Lévy input noise $\dot{N}$ renders the OU output process ξ both *stationary* and *Markov*. On the other hand, in the case of MA dynamics a Lévy input noise $\dot{N}$ renders the MA output process Ξ *stationary*, but does not render it Markov (albeit when the memory function $\Phi(\tau)$ is exponential).

Thus, transcending from the 'microscopic' parts-level OU dynamics to the 'macroscopic' system-level MA dynamics induces a *Markov-breaking* phenomenon: while the parts' output processes ξ_k are Markov, the system's output process Ξ is not. The Markov breaking phenomenon, in turn, may further lead to the emergence of a *long memory* [33–35]—or, as coined by Mandelbrot and Wallis, the emergence of a '*Joseph effect*' [36].

A finite-variance stationary stochastic process is said to have a long memory [33–35] if either of the following equivalent asymptotic conditions holds: (i) its auto-covariance function $R(t)$ admits the asymptotic form

$$R(t) \underset{|t|\to\infty}{\sim} \frac{r(|t|)}{|t|^{\alpha}}, \tag{7}$$

where the exponent α is in the range $0 < \alpha < 1$, and where the function $r(\tau)$ $(\tau > 0)$ is slowly varying at infinity; (ii) its power-spectrum function $S(\omega)$ admits the asymptotic form

$$S(\omega) \underset{\omega\to 0}{\sim} \frac{s(|\omega|)}{|\omega|^{\beta}}, \tag{8}$$

where the exponent β is in the range $0 < \beta < 1$, and where the function $s(\tau)$ $(\tau > 0)$ is slowly varying at the origin[4].

(The equivalence of the conditions represented by equations (7) and (8) follows from the Tauberian theorem 4.10.3 in [37]—which asserts that the exponents α and β satisfy the connection $\alpha + \beta = 1$, and provides explicit transformations between the slowly varying functions $r(\tau)$ and $s(\tau)$.)

[4] A real function $\varphi(\tau)$ $(\tau > 0)$ is said to be *slowly varying* at the limit point τ_* if the limit $\lim_{\tau\to\tau_*} \varphi(c\tau)/\varphi(\tau) = 1$ holds for all positive constants c ([37, 38], section XIII.5). A slowly varying function fluctuates slower than a power-law (at the limit point τ_*). Examples include constant functions, logarithms, iterated logarithms and powers of logarithms.

Both the OU process ξ of equation (2) and the MA process Ξ of equation (5) are of finite variance if and only if the driving Lévy noise $\dot{N}$ is of finite variance. With no loss of generality, we henceforth consider the Lévy noise $\dot{N}$ to be centered—i.e., to have zero mean and unit variance. This implies that both processes ξ and Ξ have zero mean, and have well-defined auto-covariance functions.

The OU process ξ can never display a long memory, since its auto-covariance function always admits an exponential decay. On the other hand, the MA process Ξ can display a long memory, depending on its memory function $\Phi(\tau)$—which, in turn, is contingent on the distribution of the random amplitude-pair (X, Y). Hence, the emergence of a long memory—in the transcendence from the microscopic parts-level OU dynamics to the macroscopic system-level MA dynamics—is determined solely by the system's quenched randomness, and is unaffected by the system's annealed randomness.

Let $\psi_{X,Y}(x, y)$ $(x, y > 0)$ denote the probability density function of the random amplitude-pair (X, Y). Long memory of the MA process Ξ turns out to be contingent on the behavior of the *base function* $\Psi(x) = \int_0^\infty y\psi_{X,Y}(x, y)\,\mathrm{d}y$ $(x > 0)$ near the origin. Specifically, the emergence of long memory is characterized as follows [32].

If the base function satisfies $\Psi(x) \sim \varphi(x)x^{1/2-H}$ (as $x \to 0$)—where the exponent H is in the range $1/2 < H < 1$, and the function $\varphi(x)$ is slowly varying at the origin—then the MA process Ξ has a long memory: (i) equation (7) holds with exponent $\alpha = 2 - 2H$ and slowly varying function $r(\tau) \sim \varphi(1/\tau)^2$ (as $\tau \to \infty$); (ii) equation (8) holds with exponent $\beta = 2H - 1$ and slowly varying function $s(\tau) \sim \varphi(\tau)^2$ (as $\tau \to 0$).

5. Examples

Two antithetical scenarios of amplitude-pair distribution are independent amplitudes and functionally-dependent amplitudes $Y = f(X)$.

Let $\psi_X(x)$ $(x > 0)$ denote the probability density function of the random amplitude X. In the 'independent scenario' the memory function admits the form $\Phi(\tau) = \langle Y\rangle\langle\exp\{-\tau X\}\rangle$, and the base function admits the form $\Psi(x) = \langle Y\rangle\psi_X(x)$. In the 'functionally-dependent scenario' the memory function admits the form $\Phi(\tau) = \langle f(X)\exp\{-\tau X\}\rangle$, and the base function admits the form $\Psi(x) = f(x)\psi_X(x)$.

A special case of the COU system-model—in which the amplitude X is $Gamma(\gamma)$-distributed[5], the amplitude Y is degenerate and deterministic (specifically, $Y \equiv 1$), and the input noise $\dot{N}$ is white—was studied by Iglói and Terdik [29]. More generally, if the 'independent scenario' holds and the amplitude X is $Gamma(\gamma)$-distributed then the memory function admits the Paretian form $\Phi(\tau) = \langle Y\rangle(1 + \tau)^{-\gamma}$, and the base function satisfies $\Psi(x) \sim x^{\gamma-1}$ (as $x \to 0$). Long memory emerges when the exponent γ is in the range $1/2 < \gamma < 1$, in which case $H = 3/2 - \gamma$.

If the 'functionally-dependent scenario' holds with power-law functional dependence $Y = X^\delta$, and the amplitude X is $Gamma(\gamma)$-distributed then (the exponents γ and δ need satisfy $\gamma + \delta > 0$): the memory function admits the Paretian form $\Phi(\tau) = \frac{\Gamma(\gamma+\delta)}{\Gamma(\gamma)}(1+\tau)^{-(\gamma+\delta)}$, and the base function satisfies $\Psi(x) \sim x^{(\gamma+\delta)-1}$ (as $x \to 0$). Long memory emerges when the exponent $\gamma + \delta$ is in the range $1/2 < \gamma + \delta < 1$, in which case $H = 3/2 - (\gamma + \delta)$. The special case $\delta = 0$ corresponds to $Y \equiv 1$, and the special case $\delta = 1/2$ corresponds to the fluctuation–dissipation theorem [31]. In the latter 'fluctuation–dissipation case' a

[5] Namely, the random variable X is governed by the probability density function $\psi_X(x) = \Gamma(\gamma)^{-1}\exp\{-x\}x^{\gamma-1}$ $(x > 0)$, where γ is an arbitrary positive parameter ([38], section II.2).

long memory emerges when the exponent γ is in the range $0 < \gamma < 1/2$, in which case $H = 1 - \gamma$.

6. Fractional Brownian motion

We turn now to explore the *temporal scaling* of the MA process Ξ in the presence of *long memory* ($1/2 < H < 1$). Speeding up time by the positive factor a, while re-scaling the process Ξ by the positive factor b, yields the scaled MA process $V_{a,b} = (V_{a,b}(t))_t$ given by $V_{a,b}(t) = b^{-1}\Xi(at)$.

As the scaling factors grow to infinity ($a, b \to \infty$)—whilst satisfying the scaling relation $b \sim \sqrt{a}\Phi(a)$—the scaled MA process $V_{a,b}$ converges stochastically to a limit noise process $V = (V(t))_t$ given formally by

$$V(t) = \int_{-\infty}^{t} (t - t')^{H-3/2}\dot{W}(t')\,\mathrm{d}t', \qquad (9)$$

where $\dot{W} = (\dot{W}(t))_t$ is a driving white noise [32]. Namely, the process V is a temporal moving-average of white noise $\dot{W}$, and the temporal averaging is performed with respect to the power-law memory function $\tau^{H-3/2}$ ($\tau > 0$; $1/2 < H < 1$).

The limit V is a noise process: it is the temporal derivative of FBM in the very same way white noise is the temporal derivative of Brownian motion. Specifically, if we take the limiting noise process V to be a *random velocity*, then the resulting motion $M = (M(t))_{t\geqslant 0}$—defined by $M(t) = \int_0^t V(t')\,\mathrm{d}t'$—is FBM.

FBM—a generalization of Brownian motion first introduced by Mandelbrot and Van Ness [3]—is the quintessential example of a random motion with continuous sample-path trajectories and dependent increments. By definition [4], FBM is a zero-mean *Gaussian process* governed (up to a multiplicative factor) by the auto-covariance function

$$\langle M(t_1)M(t_2)\rangle = \tfrac{1}{2}\{|t_1|^{2H} + |t_2|^{2H} - |t_1 - t_2|^{2H}\}. \qquad (10)$$

The parameter H is the FBM's *Hurst exponent*, and can assume values in the range $0 < H < 1$—the case $H = 1/2$ corresponding to Brownian motion. FBM is a statistically *selfsimilar* process [4]: for any positive constant c the re-scaled motion $(c^{-H}M(ct))_{t\geqslant 0}$ is equal, in law, to the 'original' motion $(M(t))_{t\geqslant 0}$.

The mean-square displacement of FBM is given by $\langle M(t)^2\rangle = t^{2H}$—which is sub-diffusive in the sub-range $0 < H < 1/2$, diffusive when $H = 1/2$, and super-diffusive in the sub-range $1/2 < H < 1$ (which is our case).

The increments of FBM form a finite-variance stationary stochastic sequence. This sequence is negatively correlated in the sub-range $0 < H < 1/2$, uncorrelated when $H = 1/2$, and positively correlated in the sub-range $1/2 < H < 1$. Moreover, in our case—the sub-range $1/2 < H < 1$—the sequence of increments possesses a long memory: (i) equation (7) holds with exponent $\alpha = 2 - 2H$ and slowly varying function $r(\tau) \sim 1$ (as $\tau \to \infty$); (ii) equation (8) holds with exponent $\beta = 2H - 1$ and slowly varying function $s(\tau) \sim 1$ (as $\tau \to 0$).

7. Conclusions

In this communication we presented a path—based on the COU system-model—leading from OU dynamics to MA dynamics and thereafter to FBM. The path exploited two scaling limits: *structural* and *temporal*.

The structural scaling limit 'zoomed out' on the structure of COU systems—transcending from the microscopic parts-level to the macroscopic system-level. The structural scaling limit had two key effects: (i) the dynamics changed from OU to MA; (ii) the quenched randomness 'solidified' into the deterministic memory function $\Phi(\tau)$ of the MA dynamics.

The temporal scaling limit 'zoomed out' on the MA dynamics time series and—in the presence of long memory—yielded FBM. The temporal scaling limit turned out to be *universal*: (i) the quenched randomness collapsed to a single one-dimensional parameter—the Hurst exponent H governing the FBM's process distribution; (ii) the annealed randomness—the arbitrary centered Lévy statistics of the driving noise—became Gaussian.

The change of dynamics from OU to MA was accompanied by a Markov-breaking phenomenon which, in turn, enabled the emergence of a long memory. The COU system presented herein—constructed from simple and elementary OU 'building blocks'—is a tractable analytical model providing both a qualitative explanation and a quantitative characterization of the Markov-breaking phenomenon and the emergence of long memory.

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